A METHOD OF APPROACH TO PROBLEMS OF THE THEORY OF HEAT CONDUCTION, DIFFUSION AND THE WAVE THEORY AND OTHER SIMILAR PROBLEMS IN PRESENCE OF MOVING BOUNDARIES AND ITS APPLICATIONS TO OTHER PROBLEMS

(OB ODNOM VOZMOZHNOM METODE PODKHODA K RASSMOTRENIIU ZADACH TEORII TEPLOPROVODNOSTI, DIFFUSII, VOLNOVYKH I IM PODOBNYKH PRI NALICHII DVIZHUSHCHIKHSIA GRANITS I O NEKOTORYKH INYKH EGO PRILOZHENIIAKH)

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G. A. GRINBERG
(Leningrad)

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Range of problems in which the heat conduction, diffusion, wave and other similar equations are encountered and have to be solved over the regions whose shape is time-dependent, is very large. It includes the cases when the motion of boundaries is given and more complex cases when the motion has to be determined from the conditions of the problem.

First kind of problem is met in soil mechanics, dam building theory, etc. [1 and 2], while the other kind arises in connection with problems of melting and freezing of materials, diffusion processes involving phase changes e. a. In the latter case, processes taking place in two or more adjacent media have to be considered simultaneously and the law describing the displacement of phase boundaries has to be found from the conditions of the problem [3 and 4]. Some problems of electrodynamics occupy the intermediate position, namely those connected with generation of very intense magnetic fields by means of rapid compression of a magnetic flow. These may fall into the first or second category, depending on whether the motion of boundaries is assumed given, or has to be determined from the conditions of the problem [5 and 6].

Exact solutions of problems of this type are, on the whole, obtained by trial and error, are known only in a limited number of cases, and usually for a narrow set of initial conditions (*). More general approach to some of these problems was achieved by the use of methods based on integral or integro-differential equations and of numerical methods (see the bibliography in [3, 4 and 7]).

This paper presents a basically different method of approach to solving some classes of such problems. The method is based on expansion of solutions into series in terms of sets of "instantaneous" eigenfunctions of the corresponding problems. Below we consider problems of the first kind, i. e. problems in which the motion of boundaries is assumed given. It should however be noted that the results obtained can be utilized in solving

*) Such as, for example, solution of the fundamental Stefan's problem on freezing of a liquid and its various generalizations. See [3 and 4].

problems of the second kind both in their exact form and as a starting point for the method of consecutive approximations. This is due to the fact that in problems of the second type, the law obeyed by the moving boundaries can usually be deduced from physical considerations and without the actual knowledge of solution of the problrm (see e. g. [3], p. 276, Section 1). We should also note that the proposed method can also be used to solve certain classes of static problems, in which case the "instantaneous" eigenfunctions are replaced by "local" eigenfunctions.

1. We shall begin by considering a simple one-dimensional problem in which u is dependent on a single Cartesian coordinate x and on time t. Then, the basic diffusion equation or equation of heat conduction, has the form

$$\partial u / \partial t = \varkappa^2 \partial^2 u / \partial x^2 + q(x, t) \tag{1.1}$$

where $\varkappa$ is a constant, while $q(x, t)$ is a given function of x and t.

Let us seek the solution of (1.1) in case when the process under consideration takes place in the region whose boundaries $x = \xi_1(t)$ and $x = \xi_2(t)$ move, in general, with time. Boundary conditions are given on them and are of the first or second kind, i. e. they either are the values of

$$u|_{x=\xi_1} = f_1(t), \qquad u|_{x=\xi_2} = f_2(t)$$

or of normal derivatives

$$\partial u / \partial x|_{x=\xi_1} = \varphi_1(t), \qquad \partial u / \partial x|_{x=\xi_2} = \varphi_2(t)$$

where $f_1(t)$, $f_2(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ are given functions of time. In addition, we have known initial conditions

$$u|_{t=0} = F(x), \qquad \xi_1(0) \leqslant x \leqslant \xi_2(0)$$

If boundaries were fixed, i. e. if ξ_1 and ξ_2 were independent of time, then solution of the problem could be sought in the form

$$u = \sum_{k=0}^{\infty} U_k^{(i)}(t)\, v_k^{(i)}(x) \quad (i = 1, 2) \left(v_k^{(1)} = \sin\frac{\pi k (x - \xi_1)}{\xi_2 - \xi_1}, \; v_k^{(2)} = \cos\frac{\pi k (x - \xi_1)}{\xi_2 - \xi_1}\right) \tag{1.2}$$

Here $v_k^{(i)}(x)$ denote the eigenfunctions of the problem, which are equal to $v_k^{(1)}$ for the boundary conditions of first kind and to $v_k^{(2)}$ for the boundary conditions of second kind (*). Coefficients $U_k^{(i)}(t)$ of the expansion are given by

$$U_k^{(i)}(t) = \int_{\xi_1}^{\xi_2} v_k^{(i)}(x)\, u(x, t)\, dx \left(\int_{\xi_1}^{\xi_2} v_k^{(i)2}(x)\, dx\right)^{-1} = \frac{2}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} u(x, t)\, v_k^{(i)}(x)\, dx$$
$$(i = 1, 2) \tag{1.3}$$

and can be found from an ordinary linear first order differential equation obtained by multiplying (1.1) by $v_k^{(i)}(x)\, dx$, integrating the resulting expression in x from ξ_1 to ξ_2 and utilizing the corresponding boundary and initial conditions (see e. g. [8], Section 21).

Let now $\xi_1 = \xi_1(t)$ and $\xi_2 = \xi_2(t)$ where $\xi_1(t)$ and $\xi_2(t) > \xi_1(t)$ are some given functions of t. Then (and this is the essence of our method), we shall seek the solution of our problem in the same form (1.2) as in case of fixed boundaries, i. e. as

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*) They satisfy the following boundary conditions

$$v_k^{(1)}(\xi_1) = v_k^{(1)}(\xi_2) = 0 \text{ and } v_k^{(2)\prime}(\xi_1) = v_k^{(2)\prime}(\xi_2) = 0.$$

$$u = \sum_{k=1}^{\infty} U_k^{(1)}(t) \sin \frac{\pi k (x-\xi_1)}{\xi_2 - \xi_1} \tag{1.4}$$

for the boundary conditions of first kind when

$$u|_{x=\xi_1(t)} = f_1(t), \qquad u|_{x=\xi_2(t)} = f_2(t) \tag{1.5}$$

or as

$$u = \sum_{k=0}^{\infty} U_k^{(2)}(t) \cos \frac{\pi k (x-\xi_1)}{\xi_2 - \xi_1} \tag{1.6}$$

for the boundary conditions of the second kind

$$\left.\frac{\partial u}{\partial x}\right|_{x=\xi_1(t)} = \varphi_1(t), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=\xi_2} = \varphi_2(t) \tag{1.7}$$

with the understanding that here ξ_1 and ξ_2 are no longer constants, but instead known functions of time. In other words, $U_k^{(i)}(t)$ are the coefficents of the expansion of function u which is to be determined, at any instant of time, in terms of functions

$$v_k^{(1)}(x, t) = \sin \frac{\pi k (x - \xi_1)}{\xi} \quad \text{or} \quad v_k^{(2)}(x, t) = \cos \frac{\pi k (x-\xi_1)}{\xi} \qquad (\xi = \xi_2 - \xi_1)$$

which we shall call "instantaneous" eigenfunctions (*) corresponding to the instant t. It will be an essential requirement that these coefficients $U_k^{(i)}(t)$ are, as before, given by Formula (1.3) in which $v_k^{(i)}$ are no longer dependent on x only, but also depend on t. Introducing the expression

$$w_k^{(i)}(t) = \int_{\xi_1(t)}^{\xi_2(t)} w(x, t)\, v_k^{(i)}(x, t)\, dx \tag{1.8}$$

where $w(x,t)$ is an arbitrary function of x and t, we can now write the formula for u as

$$u = \frac{2}{\xi} \sum_{k=0}^{\infty} u_k^{(i)}(t)\, v_k^{(i)}(x, t) \qquad (i = 1, 2) \tag{1.9}$$

To find $u_k^{(i)}(t)$ we shall multiply, as in case of constant ξ_1 and ξ_2, Equation (1.1) by $v_k^{(i)}(x, t)$ and integrate the resulting expression with respect to x, from $\xi_1(t)$ to $\xi_2(t)$. This gives

$$\int_{\xi_1}^{\xi_2} v_k^{(i)}(x, t) \frac{\partial u}{\partial t} dx = \varkappa^2 \int_{\xi_1}^{\xi_2} v_k^{(i)}(x, t) \frac{\partial^2 u}{\partial x^2} dx + q_k^{(i)}(t) \tag{1.10}$$

i. e. if we take into account the fact that

$$\int_{\xi_1(t)}^{\xi_2(t)} a_k^{(i)}(x, t) \frac{\partial u}{\partial t} dx = \frac{d}{dt} \int_{\xi_1(t)}^{\xi_2(t)} v_k^{(i)}(x, t)\, u\, dx - \int_{\xi_1(t)}^{\xi_2(t)} u \frac{\partial v_k^{(i)}}{\partial t} dx -$$
$$- \dot{\xi}_2 v_k^{(i)}(\xi_2, t)\, u(\xi_2, t) + \dot{\xi}_1 v_k^{(i)}(\xi_1, t)\, u(\xi_1, t) \quad (\dot{\xi}_1 = d\xi_1/dt \text{ etc.}) \tag{1.11}$$

$$\int_{\xi_1(t)}^{\xi_2(t)} v_k^{(i)}(x, t) \frac{\partial^2 u}{\partial x^2} dx = \left(v_k^{(i)} \frac{\partial u}{\partial x} - \frac{\partial v_k^{(i)}}{\partial x} u \right) \Bigg|_{x=\xi_1(t)}^{x=\xi_2(t)} + \int_{\xi_1}^{\xi_2} \frac{\partial^2 v_k^{(i)}}{\partial x^2} u\, dx =$$
$$= \left(v_k^{(i)} \frac{\partial u}{\partial x} - \frac{\partial v_k^{(i)}}{\partial x} u \right) \Bigg|_{x=\xi_1}^{x=\xi_2} - \left(\frac{\pi k}{\xi} \right)^2 u_k^{(i)} \tag{1.12}$$

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*) They are the very eigenfunctions in which u would have to be expanded into series (1.2), had the motion of boundaries terminated at that instant.

the latter being true for $\xi = \xi(t)$ by virtue of

$$\frac{\partial^2 v_k^{(i)}}{\partial x^2} = -\left(\frac{\pi k}{\xi}\right)^2 v_k^{(i)}$$

we obtain

$$\frac{du_k^{(i)}}{dt} + \left(\frac{\pi k \varkappa}{\xi}\right)^2 u_k^{(i)} = \varkappa^2 \left(v_k^{(i)} \frac{\partial u}{\partial x} - \frac{\partial v_k^{(i)}}{\partial x} u\right)\Big|_{\xi_1}^{\xi_2} + q_k^{(i)}(t) +$$

$$+ \dot{\xi}_2 v_k^{(i)}(\xi_2, t)\, u(\xi_2, t) - \dot{\xi}_1 v_k^{(i)}(\xi_1, t)\, u(\xi_1, t) + \int_{\xi_1}^{\xi_2} \frac{\partial v_k^{(i)}}{\partial t} u\, dx \qquad (1.13)$$

which together with (1. 5), yields

$$\frac{du_k^{(1)}}{dt} + \left(\frac{\pi k \varkappa}{\xi}\right)^2 u_k^{(1)} = \frac{\pi k \varkappa^2}{\xi}\left[f_1(t) + (-1)^{k-1} f_2(t)\right] + q_k^{(1)}(t) -$$

$$- \frac{\pi k}{\xi^2}\left\{\dot{\xi}\int_{\xi_1}^{\xi_2} (x - \xi_1)\, u \cos\frac{\pi k (x-\xi_1)}{\xi} dx + \dot{\xi}_1 \xi \int_{\xi_1}^{\xi_2} u \cos\frac{\pi k (x-\xi_1)}{\xi} dx\right\} \qquad (1.14)$$

for boundary conditions of the first kind, and

$$\frac{du_k^{(2)}}{dt} + \left(\frac{\pi k \varkappa}{\xi}\right)^2 u_k^{(2)} = \varkappa^2\left[(-1)^k \varphi_1(t) - \varphi_2(t)\right] + q_k^{(2)}(t) + (-1)^k \dot{\xi}_2\, u(\xi_2, t) - \qquad (1.15)$$

$$- \dot{\xi}_1 u(\xi_1, t) + \frac{\pi k}{\xi^2}\left\{\dot{\xi}\int_{\xi_1}^{\xi_2} (x - \xi_1)\, u \sin\frac{\pi k (x-\xi_1)}{\xi} dx + \dot{\xi}_1 \xi \int_{\xi_1}^{\xi_2} u \sin\frac{\pi k (x-\xi_1)}{\xi} dx\right\}$$

for boundary conditions of the second kind.

Utilizing Formula (1. 9) to compute the integrals occurring in (1. 14) and (1. 15), we finally find

$$\frac{du_k^{(1)}}{dt} + \left(\frac{\pi k \varkappa}{\xi}\right)^2 u_k^{(1)} = \frac{\pi k \varkappa^2}{\xi}\left[f_1(t) + (-1)^{k-1} f_2(t)\right] + q_k^{(1)}(t) + \qquad (1.16)$$

$$+ \frac{k}{\xi}\sum_{m=1}^{\infty}\left[(-1)^{m+k}\dot{\xi}_2 - \dot{\xi}_1\right] m \beta_{km} u_m^{(1)}(t)$$

$$\frac{du_k^{(2)}}{dt} + \left(\frac{\pi k \varkappa}{\xi}\right)^2 u_k^{(2)} = \varkappa^2\left[(-1)^k \varphi_1(t) - \varphi_2(t)\right] + q_k^{(2)}(t) + \qquad (1.17)$$

$$+ \frac{1}{\xi}\sum_{m=0}^{\infty}\left[(-1)^{m+k}\dot{\xi}_2 - \dot{\xi}_1\right](k^2\beta_{km} + 2)\, u_m^{(2)}(t)$$

$$\beta_{km} = \frac{2}{m^2 - k^2} \quad \text{for } m \neq k, \qquad \beta_{km} = \frac{1}{2k^2} \quad \text{for } m = k \qquad (1.18)$$

The initial condition $u|_{t=0} = F(x)$, $\xi_1(0) \leqslant x \leqslant \xi_2(0)$ from which it follows that

$$u_k^{(i)}(0) = \int_{\xi_1(0)}^{\xi_2(0)} v_k^{(i)}(x, 0)\, u|_{t=0}\, dx = \int_{\xi_1(0)}^{\xi_2(0)} v_k^{(i)}(x, 0)\, F(x)\, dx \qquad (1.19)$$

now enables us to obtain, for functions $u_k^{(i)}(t)$ an infinite set of combined linear first order differential equations, which yield these functions, provided that initial values (1. 19) are known.

If both boundaries are stationary, then all equations for separate $u_k^{(i)}$ become independent of each other and solution can be obtained in terms of quadratures (see [8], pp. 207-210 and following).

2. Let us, for the sake of completeness, consider the case of boundary conditions of the third kind, when

$$\left(\frac{\partial u}{\partial x} - \lambda u\right)\Big|_{x=\xi_1(t)} = \psi_1(t), \quad \left(\frac{\partial u}{\partial x} + \mu u\right)\Big|_{x=\xi_2(t)} = \psi_2(t) \tag{2.1}$$

where $\psi_i(t)$ ($i = 1, 2$) are given functions, while λ and μ are either constants or some known functions of t (in real physical problems we usually have $\lambda > 0$, $\mu > 0$).

Let us introduce the following complete system (see [8], pp. 81-85) of orthonormalized functions $v_k^{(3)}$:

$$v_k^{(3)}(x, t) = A_k\left[\xi\lambda \sin\frac{\gamma_k(x-\xi_1)}{\xi} + \gamma_k \cos\frac{\gamma_k(x-\xi_1)}{\xi}\right] \tag{2.2}$$

where $\gamma_k = \gamma_k(t)$ are the roots of

$$\cot \gamma_k = \frac{1}{\xi(\lambda+\mu)}\left[\gamma_k - \frac{\lambda\mu\xi^2}{\gamma_k}\right] \qquad (k = 1, 2, 3, \ldots, \infty) \tag{2.3}$$

Quantities $A_k = A_k(t)$ are found from the normalizing condition

$$\int_{\xi_1}^{\xi_2} v_k^{(3)2}(x, t)\, dx = 1 \tag{2.4}$$

These functions satisfy the differential equation

$$\partial^2 v_k^{(3)} / \partial x^2 = -\gamma_k{}^2 v_k^{(3)} / \xi^2 \tag{2.5}$$

and boundary conditions

$$\frac{\partial v_k^{(3)}}{\partial x}\Big|_{x=\xi_1} = \lambda v_k^{(3)}(\xi_1, t), \qquad \frac{\partial v_k^{(3)}}{\partial x}\Big|_{x=\xi_2} = -\mu v_k^{(3)}(\xi_2, t) \tag{2.6}$$

Assuming

$$u(x, t) = \sum_{k=1}^{\infty} u_k^{(3)}(t)\, v_k^{(3)}(x, t) \tag{2.7}$$

extending Expression (1.8) to the value $i = 3$ and utilizing the relations (1.10) to (1.12) which are still valid for $i = 3$ provided that in (1.12) the term $(\pi k/\xi)^2 u_k^{(i)}$ is replaced with $(\gamma_k/\xi)^2 u_k^{(3)}$ we obtain, for $u_k^{(3)}$, Equation

$$\frac{du_k^{(3)}}{dt} + \frac{\gamma_k^2\varkappa^2}{\xi^2} u_k^{(3)} = \varkappa^2\left[v_k^{(3)}\frac{\partial u}{\partial x} - \frac{\partial v_k^{(3)}}{\partial x} u\right]\Big|_{\xi_1}^{\xi_2} + q_k^{(3)}(t) +$$
$$+ \dot{\xi}_2 v_k^{(3)}(\xi_2, t)\, u(\xi_2, t) - \dot{\xi}_1 v_k^{(3)}(\xi_1, t)\, u(\xi_1, t) + \int_{\xi_1}^{\xi_2} \frac{\partial v_k^{(3)}}{\partial t} u\, dx \tag{2.8}$$

i. e. with (2.6), (2.7) and (2.1) taken into account,

$$\frac{du_k^{(3)}}{dt} + \frac{\gamma_k^2\varkappa^2}{\xi^2} u_k^{(3)} = \varkappa^2 [v_k^{(3)}(\xi_2, t)\,\psi_2(t) - v_k^{(3)}(\xi_1, t)\,\psi_1(t)] +$$
$$+ q_k^{(3)}(t) + \sum_{m=1}^{\infty} \delta_{km}(t)\, u_m^{(3)}(t) \tag{2.9}$$

$$\delta_{km}(t) = \dot{\xi}_2 v_k^{(3)}(\xi_2, t)\, v_m^{(3)}(\xi_2, t) - \dot{\xi}_1 v_k^{(3)}(\xi_1, t)\, v_m^{(3)}(\xi_1, t) +$$
$$+ \int_{\xi_1}^{\xi_2} v_m^{(3)}(x, t)\, \frac{\partial v_k^{(3)}}{\partial t} dx$$

The latter, together with the initial condition (1.19) in which we put $i=3$, constitutes an infinite set of combined differential equations with given initial conditions, for the coefficients of $u_k^{(3)}(t)$.

3. Until now, we have only considered plane one-dimensional problems. It is easily seen that the proposed method can be extended to more complex cases, e.g. diffusion or thermal processes with cylindrical or spherical symmetry, and some more general cases. The basic idea of seeking the solution of the problem with moving boundaries in terms of expansion in eigenfunctions of the corresponding problem with stationary boundaries which, at any instant of time, coincide with the position of actual (moving) boundaries ("instantaneous" system of eigenfunctions of the problem) and of setting up equations defining the coefficients of this expansion, is maintained.

Let us consider, for example, such problems in case of boundary conditions of the first kind (*). Basic equation of the problem has the form

$$\frac{\partial u}{\partial t} = \varkappa^2 \frac{1}{r^n} \frac{\partial}{\partial r}\left(r^n \frac{\partial u}{\partial r}\right) + q(r, t) \qquad (0 \leqslant r_1(t) < r < r_2(t)) \qquad (3.1)$$

where $n=1$ in the cylindrical and $n=2$ in the spherical case, while $r_1(t)$, $r_2(t)$ and $q(r, t)$ are given functions.

We first consider the spherical case. Let the boundary condition be

$$u|_{r=r_1(t)} = f_1(t), \qquad u|_{r=r_2(t)} = f_2(t) \qquad (3.2)$$

and let

$$u|_{t=0} = F(r) \qquad (r_1(0) \leqslant r \leqslant r_2(0)) \qquad (3.3)$$

be the initial conditions.

Introducing a new function $w = ru$, we can write (3.1) as

$$\frac{\partial w}{\partial t} = \varkappa^2 \frac{\partial^2 w}{\partial r^2} + rq(r, t) \qquad (3.4)$$

Boundary and initial conditions will now become

$$w|_{r=r_1(t)} = r_1(t) f_1(t), \qquad w|_{r=r_2(t)} = r_2(t) f_2(t) \qquad (3.5)$$

$$w|_{t=0} = rF(r) \qquad (r_1(0) \leqslant r \leqslant r_2(0)) \qquad (3.6)$$

and this reduces the problem to the plane (**) case already discussed in Section 1.

Let us now consider the cylindrical case. We shall, for simplicity, consider the solid cylinder only ($r_1 = 0$) (the case of a hollow cylinder, i.e. $r_1(t) > 0$ is as easy and will not be considered here). Equations of the problem will, under the additional assumption that $r_2(t) = R(t)$, be

$$\frac{\partial u}{\partial t} = \frac{\varkappa^2}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + q(r, t) \qquad (3.7)$$

$$u|_{r=R(t)} = f(t), \qquad u|_{t=0} = F(r) \qquad (0 \leqslant r \leqslant R(0)) \qquad (3.8)$$

Eigenfunctions $v(r, t)$ of the corresponding cylindrical problem for the case when the outer radius R of the region is assumed constant are well known, and equal to (***)

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*) In case of boundary conditions of the second or third kind, the procedure is analogous to that adopted in Sections 1 and 2, in the plane case.

**) Boundary conditions for u of the second or third kind would, in general, reduce to conditions of the third kind for w, i.e. to the case discussed in Section 2.

***) Cf. [8] 21.5, p. 213. Eigenfunctions for the hollow cylinder i.e. for the region $0 < r_1 < r_2$, can be found in e.g. [3].

$$v_k(r, R) = J_0(rx_k/R) \qquad (k = 1, 2, 3, \ldots, \infty) \tag{3.9}$$

where $J_0(x)$ is the Bessel function of zero order, x_k are positive roots of Equation $J_0(x) = 0$ and R is a constant parameter.

Expansion of the arbitrary function $\varphi(r)$ in terms of functions $v_k(r, R)$ has the form

$$\varphi(r) = \frac{2}{R^2} \sum_{k=1}^{\infty} \frac{\varphi_k}{J_1^2(x_k)} \qquad \left(\varphi_k = \int_0^R r\varphi(r) v_k \, dr\right) \tag{3.10}$$

As we said before, solution of the problem when $R = R(t)$, will be sought in the same form, replacing however R in (3.9) and (3.10) with a corresponding function of time (e.g. $v_k = v_k[r, R(t)]$, etc.). To obtain the equations for

$$u_k = u_k(t) = \int_0^{R(t)} ruv_k[r, R(t)] \, dr \tag{3.11}$$

we shall again, as in the case R = const, multiply

$$\frac{\partial u}{\partial t} = \frac{\varkappa^2}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + q(r, t) \tag{3.12}$$

by $rv_k[r, R(t)]dr$ and integrate the result over the limits $[0, R(t)]$ (cf. [8], 21.5).

Now, procedure analogous to that used in derivation of (1.14), leads to the relation

$$\frac{du_k}{dt} + \left(\frac{x_k \varkappa}{R}\right)^2 u_k = \varkappa^2 x_k J_1(x_k) f(t) + q_k(t) + \frac{\dot{R}}{R} \sum_{m=1}^{\infty} \lambda_{km} u_m \tag{3.13}$$

$$\lambda_{km} = \frac{2x_k J_1(x_k)}{J_1(x_m)} \mu_{km} \quad \left(\mu_{km} = \frac{x_m}{x_m^2 - x_k^2} \quad \text{for } m \neq k, \ \mu_{km} = \frac{1}{2x_k} \quad \text{for } m = k\right)$$

which we supplement with the initial condition

$$u_k(0) = \int_0^{R(0)} rF(r) J_0\left[\frac{x_k r}{R(0)}\right] dr \tag{3.14}$$

The above two equations are sufficient for solving the problem.

4. So far, we have considered problems with moving boundaries for the diffusion type equation. However, we can easily see that analogous methods can be applied to the wave problems. Suppose we require the solution of

$$\partial^2 u/\partial t^2 = \varkappa^2 \partial^2 u/\partial x^2 + q(x, t) \qquad \xi_1(t) < x < \xi_2(t) \tag{4.1}$$

where $\varkappa$ is a constant, $q(x, t)$ is a known function of x and t and where boundary conditions of the first, second and third kind are given together with initial values

$$u|_{t=0} = F_1(x), \qquad \partial u/\partial t|_{t=0} = F_2(x) \qquad (\xi_1(0) \leqslant x \leqslant \xi_2(0)) \tag{4.2}$$

for $x = \xi_1(t)$ and $x = \xi_2(t)$.

This problem is completely analogous to that considered in Sections 1 and 2 for the diffusion equation and the only difference is in the fact, that, in (4.1) a second time derivative of the unknown function appears instead of the first derivative appearing in (1.1) and, that following this, two initial conditions are given instead of one in the former case.

Solution will again be sought in form of an expansion in terms of "instantaneous" eigenfunctions of the problem, i.e. in terms of these eigenfunctions which would have to be utilized, had the boundaries ceased to move at that particular instant. Since these eigenfunctions coincide with the eigenfunctions of the corresponding diffusion problems, i.e.

with the functions $v_k^{(i)}(x, t)$ $(i = 1, 2, 3)$ introduced in Sections 1 and 2, hence, repeating the procedure used in derivation of (1. 13) to (1. 17) and (2. 8), i. e. multiplying (4. 1) by $v_k^{(i)}(x, t)$ and integrating the result from $x = \xi_1(t)$ to $x = \xi_2(t)$ we obtain, retaining the former notation, the following expression:

$$\int_{\xi_1}^{\xi_2} v_k^{(i)}(x, t) \frac{\partial^2 u}{\partial t^2} dx = \varkappa^2 \int_{\xi_1}^{\xi_2} v_k^{(i)}(x, t) \frac{\partial^2 u}{\partial x^2} dx + q_k^{(i)}(t) =$$

$$= -\frac{a_k^2 \varkappa^2}{\xi^2} u_k^{(i)} + q_k^{(i)} + \varkappa^2 \left(v_k^{(i)} \frac{\partial u}{\partial x} - \frac{\partial v_k^{(i)}}{\partial x} u \right) \Bigg|_{\xi_1}^{\xi_2} \qquad (4.3)$$

where $a_k = \pi k$ in case of boundary conditions of the first or second kind and $a_k = \gamma_k$ in case of boundary conditions of the third kind, since the right-hand sides of (1. 10) and (4. 3) are identical. We use the easily verified formula

$$\frac{d^2}{dt^2} \int_{\xi_1}^{\xi_2} u v_k^{(i)}(x, t)\, dx = \frac{d^2 u_k^{(i)}}{dt^2} = \frac{d}{dt} [\dot{\xi}_2 u(\xi_2, t) v_k^{(i)}(\xi_2, t) - \qquad (4.4)$$

$$- \dot{\xi}_1 u(\xi_1, t) v_k^{(i)}(\xi_1, t)] + \dot{\xi}_2 \left[\frac{\partial u}{\partial t} v_k^{(i)} + u \frac{\partial v_k^{(i)}}{\partial t} \right] \Bigg|_{x=\xi_2} -$$

$$- \dot{\xi}_1 \left[\frac{\partial u}{\partial t} v_k^{(i)} + u \frac{\partial v_k^{(i)}}{\partial t} \right] \Bigg|_{x=\xi_1} + 2 \int_{\xi_1}^{\xi_2} \frac{\partial u}{\partial t} \frac{\partial v_k^{(i)}}{\partial t} dx + \int_{\xi_1}^{\xi_2} u \frac{\partial^2 v_k^{(i)}}{\partial t^2} dx + \int_{\xi_1}^{\xi_2} \frac{\partial^2 u}{\partial t^2} v_k^{(i)} dx$$

to transform the left-hand side.

This equation either with (1. 9) (in case of boundary conditions of first or second kind) or with (2. 7) (in case of boundary conditions of the third kind), values of $\partial u / \partial t$ obtained from (1. 9) or (2. 7) (*) by differentiation with respect to t and the boundary conditions, together make it possible to reduce (4. 3) to

$$\frac{d^2 u_k^{(i)}}{dt^2} + \frac{a_k^2 \varkappa^2}{\xi^2} u_k^{(i)} = Q_k^{(i)}(t) + \sum_{m=0}^{\infty} (p_{km} u_m^{(i)} + q_{km} u_m^{\cdot(i)}) \quad (k = 0, 1, 2, \ldots, \infty) \qquad (4.5)$$

Here $Q_k^{(i)}(t)$ is a known function of t; p_{km} and q_{km} are known functions of ξ_1 and ξ_2 and, generally speaking, of their first and second time derivatives and of t. We supplement (4. 5) with initial conditions, first equation of (4. 2) yielding the value of $u_k^{(i)}(0)$ which, in turn, is used to obtain $u_k^{(i)\prime}(0)$ from the second equation of (4. 2). Thus, the problem reduces to finding the functions $u_k^{(i)}(t)$ from the infinite linear

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*) These series converge, in general, nonuniformly since u satisfies nonhomogeneous boundary conditions, while $v_k^{(i)}$ satisfy the corresponding homogeneous conditions. Therefore, before differentiating these series with respect to t, we must separate the part converging nonuniformly. This is easily done either by employing general rules given e. g. in [8] (Chapter XII), or by replacing u in the basic partial differential equation by

$$U = u - \left\{ f_1(t) + [f_2(t) - f_1(t)] \frac{x - \xi_1}{\xi_2 - \xi_1} \right\} \equiv u - \psi(x, t)$$

say, in case of boundary conditions of the first kind, for which we then obtain an equation of the type of (4. 1), namely

$$\frac{\partial^2 U}{\partial t^2} = \varkappa^2 \frac{\partial^2 U}{\partial x^2} + q(x, t) - \frac{\partial^2 \psi}{\partial t^2}$$

which now has homogeneous boundary conditions $U|_{\xi_1} = U|_{\xi_2} = 0$. In case of boundary conditions of the second or third kind, the treatment is analogous.

system (4.5) with known initial values of $u_k^{(i)}(t)$ and their time derivatives.

5. Entirely analogous consideration shows that, if we had to solve, instead of (4.1), corresponding wave equations for the spherically ($n = 2$) or axially ($n = 1$) symmetric problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\varkappa^2}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial u}{\partial r}\right) + q^{(n)}(r, t) \qquad 0 \leqslant r_1(t) < r < r_2(t) \tag{5.1}$$

with initial conditions

$$u\,|_{t=0} = F_1(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F_2(r) \qquad (r_1(0) \leqslant r \leqslant r_2(0)) \tag{5.2}$$

and with boundary conditions of the first, second or third kind, then, using the method adopted in solving the corresponding problems for (3.1) ("instantaneous" eigenfunctions for these problems coincide) and utilizing formulas analogous to (4.4) in transforming the integrals containing $\partial^2 u/\partial t^2$, we again arrive at a system of equations of the type (4.5) for $u_k^{(i)}(t)$, for which the initial values $\dot{u}_k^{(i)}(0)$ and $u_k^{(i)\prime}(0)$ are known.

We shall further note that the same result would be obtained, if, instead of $\partial^2 u/\partial t^2$, the left-hand sides of (4.1) or (5.1) contained expressions of the type

$$a\partial^2 u \,/\, \partial t^2 + b\partial u \,/\, \partial t + cu,$$

where a, b and c are constant coefficients.

6. Until now we were considering problems, in which diffusion, wave or other more general equations described time-dependent processes, boundaries of the region moved with time and we had to determine the course of these processes over a period of time. We can however attempt to apply these concepts to solution of other, e.g. statics problems. We shall show it first on the simplest case of the plane Poisson equation, which we shall write as

$$-\,\partial^2 u \,/\, \partial y^2 = \partial^2 u \,/\, \partial x^2 + q\,(x,\ y) \tag{6.1}$$

where $q(x, y)$ is a given function, while the unknown function $u(x, y)$ is given at two values of y, say at $y = 0$ and $y = a$ (on the segments AB and CD) and on two curves $x = x_1(y)$ and $x_2(y)$, $0 \le y \le a$ (curves AC and BD in Fig. 1). We shall seek $u(x, y)$ in form of a series

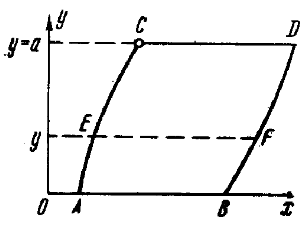

Fig. 1

$$u(x, y) = \frac{2}{\xi}\sum_{k=1}^{\infty} u_k(y)\sin\frac{\pi k\,[x - x_1(y)]}{\xi}$$

$$\begin{pmatrix} x_1(y) < x < x_2(y) \\ \xi = x_2(y) - x_1(y) \end{pmatrix} \tag{6.2}$$

Functions

$$v_k(x, y) = \sin\frac{\pi k\,[x - x_1(y)]}{\xi}$$

can be called local eigenfunctions of the problem. Using again the method employed in Section 4, Equation (4.1), i.e. multiplying (6.1) by $v_k(x, y)$ and integrating it with respect to x from $x_1(y)$ to $x_2(y)$, we obtain

$$-\int_{x_1(y)}^{x_2(y)} v_k(x, y)\frac{\partial^2 u}{\partial y^2}\,dx = -\frac{\pi^2 k^2}{\xi^2}u_k + q_k(y) +$$
$$+\frac{\pi k}{\xi}\{u\,[x_1(y), y] + (-1)^{k-1}u\,[x_2(y), y]\} \tag{6.3}$$

where the previous notation is maintained, while $u[x_i(y), y]$ ($i = 1, 2$) denote the values of $u(x, y)$ given, by definition, at the points E and F of the boundary curves (Fig. 1).

Transformation of the left-hand side in the manner similar to that used in obtaining (4. 5) from (4. 3) with t replaced by y, we arrive at a system of equations resembling (4. 5), which define the coefficients of $u_k(y)$. Boundary conditions on the segments AB and CD where the values $u(x, 0)\ (x_1(0) \leqslant x \leqslant x_2(0))$ and $u(x, a)\ (x_1(a) \leqslant \leqslant x \leqslant x_2(a))$, are given by definition, yield in addition

$$u_k(0) = \int_{x_1(0)}^{x_2(0)} u(x, 0)\, v_k(x, 0)\, dx, \qquad u_k(a) = \int_{x_1(a)}^{x_2(a)} u(x, a)\, v_k(x, a)\, dx$$
$$(k = 1, 2, 3, \ldots, \infty) \tag{6.4}$$

i. e. in the present case, functions $u_k(y)$ will have to be determined from the obtained system of linear differential equations according to their values at $y = 0$ and $y = a$ (boundary value problem).

In a similar manner we can consider, for Equation (6. 1), boundary value problems with conditions of the second or third kind and corresponding problems for axisymmetric fields defined by Laplace, Poisson and other equations.

We shall also note that although we have limited ourselves to problems with two independent variables the proposed method based on expansion of the sought solution into a series in "instantaneous" or "local" eigenfunctions of the problem can obviously be applied to the case of three or more independent variables and to symmetries other than the plane, cylindrical or spherical, provided that the basic equation of the problem allows at least partial separation of the variables.

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